4 Normable and Banach spaces

It is trivial to see that a translational-invariant metric comes from a norm if and only if it is homogeneous with respect to scalar multiplication. A bit more profound is the following theorem about the normability of a tvs topology:

Theorem 4.1 *A Hausdorff tvs V is normable if and only if V is locally convex and locally s-bounded.*

Proof. Local convexity and local s-boundedness are obviously necessary as balls are always convex and s-bounded. But the the condition is also sufficient: Take an s-bounded zero neighborhood and a convex subneighborhood U of it, then its Minkowski functional μ_U is a seminorm on V which is even a norm: Given $v \neq 0$, take any starshaped 0-neighborhood W which does not contain v. Then there is a $\lambda > 0$ with $U \subset \lambda W$, and it is easy to see that $\lambda v \notin U$. Again because of s-boundedness, the sets $n^{-1}U$, $n \in \mathbb{N}$, form a local basis. Therefore μ_U generates the topology. □

Exercise: Which tvs known up to here are normable? In particular show that the spaces $C^\infty(A, \mathbb{K})$ for A compact and $C^k(U, \mathbb{K})$ with the metric defined by a compact exhaustion are not normable.

Definition 4.2 *A map $A : V \to W$ between Banach spaces is* **bounded** *if there is a $C \in \mathbb{R}$ with $||Av|| \leq C||v||$ for all $v \in V$.*

Theorem 4.3 *A linear map between Banach spaces is continuous if and only if it is bounded.*

Proof. Consider the unit ball $B := B^1(0) \subset W$. Its preimage $A^{-1}(B)$ is an open neighborhood of 0, so it contains a ball $B_\epsilon(0)$. Using the linearity of A it is now easy to see that A is bounded by ϵ^{-1}. □

Remark: A motivation for generalizing the notion of Banach spaces towards Fréchet spaces is the fact that there is no Banach space structure on the space of smooth sections of a vector bundle such that the covariant derivative in the direction of any nonzero vector field on the base be continuous. This is seen by noting that in Banach spaces continuity is equivalent to boundedness, and by constructing sections which are eigenvectors with arbitrarily high eigenvalues of the derivative or its square (most easily seen in $C^\infty(\mathbb{S}^1, \mathbb{R})$ with $f_K(x) = K^{-1} sin(K^2 x)$).

Theorem 4.4 (Arzela-Ascoli Theorem) *Let K be a compact Hausdorff space and let $S \subset C(K) := C^0(K, \mathbb{K})$. Then $\overline{S}$ is compact if and only if S is bounded and equicontinuous.*

Proof. W.r.o.g. assume $K, S \neq \emptyset$. If $\overline{S}$ is compact, we choose $r > 0$ and $x \in K$. By compactness, S is ball-finite, so choose $f_1 ..., f_n \in S$ with $\bigcup_{i=1}^n B_{r/3}(f_i)$ covers S. By intersection of neighborhoods, let $U_{x,r}$ be a neighborhood of x with $|f_i(y) - f_i(x)| < r/3$ for all $i = 1, ..., n$ and all $y \in U_{x,r}$. Now by the usual $\epsilon/3$-argument, we have that for every $f \in S$ and $y \in U_{x,r}$, $|f(y) - f(x)| < r$, therefore S is equicontinuous (and bounded by precompactness).
Conversely, if S is bounded and equicontinuous, choose an $r > 0$ and define, for every $x \in K$, a neighorhood $U_{x,r}$ such that for every $f \in S$, $|f(y) - f(x)| < r/3$, for every $y \in U_{x,r}$. As K is compact, there are $x_1, ...x_m \in K$ with $K \subset \bigcup_{i=1}^m U_{x_i,r}$, thus $D := \{(f(x_1), ..., f(x_m)) | f \in S\}$ is a bounded subset of $\mathbb{R}^m$ in the sup metric and therefore ball-finite, so there are $f_1, ..., f_n \in S$ with $D \subset \bigcup_{i=1}^n B^{sup}_{r/3}(f_i(x_1), ..., f_i(x_m))$. Now

for every $x \in K$ and $f \in S$ there is a $k \in \{1, ..., m\}$ such that $x \in U_{x_k,r}$, and an $l \in \{1, ...n\}$ with $(f(x_1), ..., f(x_m)) \in B^{sup}_{r/3}(f_l(x_1), ...f_l(x_m))$, and with an $\epsilon/3$-argument we see easily that $f(x) - f_l(x) \leq r$ for every $x \in K$, therefore S is ball-finite and thus precompact. $\square$

Now, for a given compact topological space M and a Banach space B we consider the space $C^0(M, B)$ with the supremum norm. For $M = [0, 1]$ we write $C^0([0, 1], B) =: C$. We want to show that there is no nice differentiable function on C which is boundedly supported, i.e., with support contained in a ball. Let F, G be Banach spaces and $U \subset F$ open. We call a map $f : U \to G$ **Fréchet differentiable** iff it is continuously differentiable (cf Part 1) and if $df : U \to L(F, G)$ defined by $df(u)(x) := f'(u, x)$ takes values in $BL(F, G)$ and is continuous from U to $BL(F, G)$, the latter as usual topologized by the operator norm. The importance of Fréchet differentiability lies in its better behaviour in Inverse Function Theorems.
(For the people of the DG course: E.g., we have seen that Fréchet differentiable vector fields have a local flow while differentiable vector fields in general don't!)
We denote the space of Fréchet differentiable maps by $CF^1(U, G)$. The construction of spaces of higher Fréchet differentiability is straightforward.

Theorem 4.5 ([?], Theorem 14.9) *Let $f \in CF^1(C, \mathbb{R})$ be nontrivial. Then f is not boundedly supported.*

Proof. It is sufficient to show that for every nontrivial $f \in CF^1(C, \mathbb{R})$ with $f(0) = 0$ there is an $x \in \overline{B_2(0)} \setminus B_1(0)$ with $f(x) \leq ||x||$: indeed, if this is the case, let $f_0 \in CF^1(C, \mathbb{R})$ be nontrivial and have its support in $B_R(0)$. Then by concatenation with the multiplication by $(4R)^{-1}$ we define a nontrivial element f_1 of $CF^1(C, \mathbb{R})$ with support contained in $B_{1/4}(0)$. By choosing a point $x \in B_{1/4}(0)$ with $f_1(x) \neq 0$ and defining $t_w(v) = v + w$ we getthat $f_2 := f_1 \circ t_x \in CF^1(C) \setminus \{0\}$, $supp(f_2) \subset B_{1/2}(0)$ and $f_2(0) \neq 0$. Then $f_3 := (f_2(0))^{-1} \cdot f_2 \in CF^1(C) \setminus \{0\}$, $supp(f_3) \subset B_{1/2}(0)$, $f_3(0) = 1$, and $f_4 := 3 - 3f_3 \in CF^1(C) \setminus \{0\}$, $f_4(0) = 0$ and $f_4(x) = 3$ for all x with $||x|| \geq 1$ in contradiction to the assumption.
Our strategy will be to define iteratively a sequence $\{x_i\}_{i \in \mathbb{N}}$ in the subset $\{f(x) \leq ||x||\}$, beginning with $x_0 = 0$ about which we will show that after finitely many terms it ends up in $\overline{B_2(0)} \setminus B_1(0)$. Let $0 < \epsilon < 1$ be given. For a given x_i we define

$$U_i^\epsilon := U(x_i, \epsilon) := \{y \in C | f(y) \leq ||y||, ||y - x_i|| \leq 1, ||y|| - ||x_i|| \geq \epsilon/8 \cdot ||y - x_i||\}.$$

For every $x \in \{f(x) \leq ||x||\}$ the set $U(x_i, \epsilon)$ is nonempty as it contains x itself. Therefore $M_i^{(\epsilon)} := sup\{||y - x_i|| : y \in U_i^{(\epsilon)}\} \leq 1$ well-defined, and we choose $x_{i+1} \in U_i^{(\epsilon)}$ with $||x_{i+1} - x_i|| \geq M_i^{(\epsilon)}/2$. Now we want to show that there is an $m \in \mathbb{N}$ with $||x_m|| \geq 1$. If this is not the case, the sequence $||x_i||$ is monotonously increasing and bounded (because of the third condition for $U_i^{(\epsilon)}$, and with the same condition one sees that x_i is a Cauchy sequence, thus it has a limit L with $||L|| > 0$ (as otherwise $U_0^{(\epsilon)} = \{0\}$ and therefore $f > || \cdot ||$ in $B_1(0)$, and then f would not be differentiable in the origin: consider it along a straight line through 0). Therefore we have $0 < ||L|| \leq 1$ and $f(L) \leq ||L||$. As $f \in CF^1(C)$, there is a $\delta > 0$ with

$$f(L + u) - f(L) - df(L) \cdot u \leq \epsilon/8 \cdot ||u||$$

for all $u \in C$ with $||u|| < \delta$ (w.r.o.g. $\delta \leq 1$, $\delta \leq 2||L||$). Now we need a lemma:

Lemma 4.6 *For all $\sigma, \tau > 0$, $\tau < 1$, and for all $g \in C$, there is an $h \in B_1(0) \subset C$ with*

$$||g+th|| > ||g|| + \tau|t| - \sigma,$$

for all $|t| < ||g||$.

Proof of the lemma. Given g as above, take

$$U_\tau := f^{-1}([-||f||, -||f|| + \tau) \cup (||f|| - \tau, -||f||])$$

and choose $h_{\sigma,\tau}$ such that $\{\tau\} \cup \{-\tau\} \subset h_{\sigma,\tau}(U_\tau)$.

Now choose $g := L$, $\tau := \epsilon/2$, $\sigma := \epsilon\delta/8$ and abbreviate $h_{\sigma,t} = h$. Define $t := -sign(df(L) \cdot h) \cdot \delta/2$.
Then $|t| < ||L||$ and $L + th \in U(L, \epsilon)$ as we have the strict inequalities

$$\begin{aligned} ||L+th|| > ||L|| + \epsilon\delta/8 \geq f(L) + \epsilon/8 \cdot ||th||, \\ ||L + th - L|| = |t| \cdot ||h|| < \delta \leq 1, \\ ||L+th|| - ||L|| > \epsilon\delta/8 > \epsilon||th||/8 \end{aligned}$$

(the last inequality of the first line holds as we can omit the term containing df as we have chosen its sign correctly). As those are strict inequalities and because of continuity of f and the norm, they also hold for large x_i instead of L. Thus $M_i^{(\epsilon)} \geq ||L+th-x_i|| > \epsilon\delta/8$ by the triangle inequality and $||th|| > \epsilon\delta/8$, $||h|| > 3/4$. Therefore $||x_{i+1} - x_i|| > \epsilon\delta/16$ for all i which contradicts the Cauchy condition. $\square$

The result is transferred immediately to $C^k([0,1], B)$ by the following theorem (which is a special case of a theorem by Milutin but can be proved in a shorter way as follows):

Theorem 4.7 *Let E be a Fréchet space. For each $k \in \mathbb{N}$, there is an isomorphism of topological vector spaces between $C^0([0,1], E) =: C([0,1], E)$ and $C^k([0,1], E)$.*

Proof. First we construct an isomorphism $I_1 : C^k([0,1], E) \to E^k \times C([0,1], E)$ by $I_1(c) = (c(0), ...c^{(k-1)}(0), c^{(k)})$ where $c^{(i)}$ is the i-th derivative of c (an invese is given by the integral which is well-defined in Fréchet spaces). Now we want to show that for all $m \in \mathbb{N}$, there is a second isomorphism $I_2 : C([0,1], E) \cong E^m \times C([0,1], E)$. Let $C_0([0,1], E)$ be the subset of $C([0,1], E)$ of all elements c with $c(0) = 0$, then we have an isomorphism $i : C([0,1], E) \to C_0 E \times ([0,1], E)$ by $i(c) = (c(0), c - c(0))$. Thus it remains to construct an isomorphism $I_2 : C_0([0,1], E) \to E^m \times C_0([0,1], E)$. For a Fréchet space E, let $\rfloor(E)$ denote the Fréchet space of bounded sequences in E with the supremum topology. To this purpose, first we construct an isomorphism $J : C_0([0,1], E) \to \rfloor(E) \times \rfloor(C_{0,1}([0,1], E))$ where $C_{0,1}([0,1], E)$ means the subspace of $C([0,1], E)$ consisting of all elements c with $c(0) = 0 = c(1)$. We define J by $J(c) = (J_1(c), J_2(c))$ with $(J_1(c))_n = c(2^{-n})$ and $(J_2(c))_n(t) := c(2^{-n}t) - c(2^{-n}) \cdot t$. Then use the isomorphism $K : \rfloor(E) \to E^m \times \rfloor(E)$ given by $(K(a))_n = ((a_1, ...a_m), (a_{m+1}, ...))$. Finally put together the isomorphisms to get the desired isomorphism. $\square$

Theorem 4.8 *Let X be a Banach space, $\emptyset \neq A \subset X$ compact and convex. Then there is a continuous projection $P_A : X \to A$, $P_A \circ P_A = P_A$, with*

$$||x - P_A(x)|| = dist(x, A).$$

Proof: Exercise. □

Example: We want to construct a Schauder basis for $C^0([0,1]) =: C$. Historically, it is the first example of a Schauder basis of a non-Hilbertable space (by Schauder, 1927). The first two elements s_0 and s_1 we define by $s_0(t) := 1$, $s_1(t) := t$. For $n \geq 2$, we define s_n by first defining $m_n \in \mathbb{Z}$ s.t. $2^{m-1} < n \leq 2^m$ and then

$$s_n(t) := 2^m(t - (\frac{2n-2}{2^m} - 1)) \qquad \text{for} \qquad \frac{2n-2}{2^m} - 1 \leq t < \frac{2n-1}{2^m} - 1,$$

$$s_n(t) := 1 - 2^m(t - (\frac{2n-1}{2^m} - 1)) \qquad \text{for} \qquad \frac{2n-1}{2^m} - 1 \leq t < \frac{2n}{2^m} - 1,$$

and zero otherwise (don't worry, we will draw these functions in the exercise!). Define inductively a sequence $i \mapsto p_i \in C$ by $p_0 := f(0)s_0$,

$$\alpha_i := f(\frac{2n-1}{2^{m_n}} - 1) - p_{i-1}(\frac{2n-1}{2^{m_n}} - 1)$$

and

$$p_i := p_{i-1} + \alpha_i \cdot s_i.$$

Then p_i agrees with f in i points and interpolates linearly betwen them. It is easy to see that $p_m = \sum_{i=0}^m \alpha_i s_i$ for all m. As the interval is compact, uniform continuity of f implies that $lim_{m\to\infty}||f - p_m||_\infty = 0$, thus $f = \sum_{i\in\mathbb{N}} \alpha_n s_n$.
For uniqueness, let β_n be a sequence of real numbers with $f = \sum_{i\in\mathbb{N}} \beta_i s_i$, then $\sum_{i\in\mathbb{N}}(\alpha_i - \beta_i)s_i = 0$ and therefore $\sum(\alpha_i - \beta_i)s_i(t) = 0$ for all t of the form $\frac{2n-1}{2^{m_n}} - 1$, thus $\alpha_i = \beta_i$ for all i. Therefore the s_i form a Schauder basis for C, which is moreover normalized.

Exercise: Encuentren una base de Schauder para $L^p([0,1])$! Hay una sucesión de funciones que es una base de Schauder normalizada para *cada* L^p?

Theorem 4.9 *Let X be a Banach space with a Schauder basis $\{e_n\}$. Let P_n be the projections $x \mapsto x_n e_n$ if $x = \sum_{i\in\mathbb{N}} x_i e_i$. Then $sup_{n\in\mathbb{N}}||P_n||$ is finite.*

Proof: Exercise. □

It is easy to see that, for a Banach space space X, $B(X) := BL(X,X)$ forms a Banach algebra by pointwise addition and composition. Algebraically invertible elements of $B(X)$ have always inverse elements in $B(X)$ by the Open mapping theorem, therefore the subset $GL(X)$ of invertible elements is a subgroup. Now we want to focus on an interesting ideal of $B(X)$, the ideal of *compact operators*:

Definition 4.10 *Let X, Y be tvs. A linear map $f \in CL(X,Y)$ is called* **compact** *if $\overline{f(U)}$ is compact for every s-bounded set $U \subset X$.*

Remarks: If X is Banach, then obviously it is enough to require that the image of the unit ball be precompact. If X, Y are Banach, $A \in CL(X,Y)$ with finite-dimensional image, then A is obviously compact. Conversely:
Exercise: If X is normable and Y metrizable, $A \in B(X,Y)$ compact with closed image, then $A(X)$ is finite-dimensional.

Theorem 4.11 *Let X be a Banach space. The compact operators form a closed ideal subalgebra $CP(X)$ of $BL(X)$. Ideal means that $CP(X) \cdot BL(X), BL(X) \cdot CP(X) \subset CP(X)$.*

Proof. $CP(X)$ is a linear subspace, as multiples and sums of finite compact subsets are compact. The ideal property is by definition. The closedness comes by an $\epsilon/3$-argument: Let $A \in \overline{CP(X)}$ and $r > 0$. Then there is a $B \in CP(X)$ with $||B - A|| < r/3$. Now as $\overline{B(B_1(0))}$ is ball-finite, there are $x_1 ... x_n \in B_1(0)$ s.t. $B_{r/3}(B(x_i))$ cover $B(B_1(0))$. But then, by the triangle inequality, $B_r(B(x_i))$ cover $A(B_1(0))$. □

Theorem 4.12 *Let X, Y be Banach spaces and $f \in CL(X,Y)$. Then $f(X)$ is closed in Y if and only if there is an $r > 0$ with $||fx|| \geq r||x||$.*

Proof: Exercise. □

Theorem 4.13 *Let X be a Banach space, $A \in CP(X)$, $s \in \mathbb{K} \setminus \{0\}$. Then $ker(A - s\mathbf{1})$ is finite-dimensional.*

Proof. Restrict A to $K := ker(A - s\mathbf{1})$. The $A|_K : K \to K$ is compact, and its image K, being a kernel, is closed. Thus the theorem follows from the previous exercis. □

Theorem 4.14 *Let X be a Banach space, let $s \in \mathbb{K} \setminus \{0\}$, let $A \in BL(X)$ be compact. Then $A_s := (A - s\mathbf{1})X$ is closed.*

Proof. By Theorem 4.13, $ker(A - s\mathbf{1})$ is finite-dimensional. Let M be its topological complement. Obviously A_s is bounded linear and injective on M, with $A_s(M) = A_s(X)$. To show that $A_s(M)$ is closed, it is enough to find an $r > 0$ with $||A_s x|| \geq r||x||$ by Theorem 4.12. So if there is no such r, then there is a sequence of unit vectors x_i with $A_s(x_i) \to 0$. By compactness this contains a subsequence such that $Ax_n \to l \in X$. Therefore $sx_n \to l$, and thus $l \in M$, and $A_s l = lim(sA_s x_n) = 0$. But as S_a is injective on M, we get $l = 0$. On the other hand, $||\ sx_n|| \to s \neq 0$, a contradiction. □

Theorem 4.15 *Let Y be a non-dense subspace of a normed space X, then for every $r > 1$ there is an $x \in X$ with*

$$||x|| < r, \qquad ||x - y|| \geq 1 \qquad \forall y \in Y.$$

Proof. Choose (by scaling) an $x_0 \in X$ with $d(x_0, Y) = 1$, then choose an $y_0 \in Y$ with $||x_0 - y_0|| < r$ and define $x := x_0 - y_0$. □

Theorem 4.16 *Let X be a Banach space and $A \in BL(X)$ be compact. Then, for every $s \neq 0$ eigenvalue of A, the map $A_s := A - s\mathbf{1}$ is not surjective. Given an $r > 0$, let E_r be the set of eigenvalues s of A with $|s| \geq r$. Then E_r is finite.*

Proof. We will show that the negation of either statement implies that there are closed subspaces $Y_n \subsetneq Y_{n+1}$ and $s(n) \in \mathbb{K}$, with $A(Y_n) \subset Y_n$ and $A_{s(n+1)}(Y_{n+1}) \subset Y_n$, both for all $n \in \mathbb{N}$. Then by Theorem 4.15 we choose vectors $y_n \in Y_n$ for $n \geq 2$ with $y_n \leq 2$ and $dist(y_n, Y_{n-1}) \geq 1$. Now for $n > m \geq 2$, we define $z_{nm} := Ay_m - A_{s(n)}y_m$. Then by the conditions on the subspaces Y_m we get $z_{nm} \in Y_{n-1}$, and

$$||Ay_n - Ay_m|| = ||s_n y_n - z_{nm}|| = |s_n| \cdot ||y_n - s_n^{-1} z_{nm}|| \geq |s_n| > r.$$

Thus the sequence $n \mapsto Ay_n$ has no convergent subsequence, although it is bounded. Therefore A is not compact, a contradiction.
Now suppose that the first assertion is wrong. Then there is a $S \neq 0$ with A_S surjective. Then define $Y_n := ker(A_S^n)$. Now, as S is an eigenvalue of A, Y_1 is not empty. We choose an $y_1 \in Y_1$ and inductively y_n with $A_S y_{n+1} = y_n$, then we have automatically $y_n \in Y_n \setminus Y_{n-1}$. The second property of the Y_n holds as A_S and A commute, in the third we choose $s(i) := S$.
For the second assertion assume that there is an infinite sequence e_n of pairwise distinct eigenvalues in E_r. Then choose associated eigenvectors e_n and define define $Y_n := span(e_1, ...e_n)$. It is very easy to check that these subspaces, which are finite-dimensional and thus closed, have the required properties. □

Theorem 4.17 *Let X be a Banach space, $A \in BL(X)$ compact, and $s \in \mathbb{K} \setminus \{0\}$. Then the spaces $ker(A_s)$ and $X/A_s(X)$ have the same finite dimension.*

Proof. First we prove that $dim(ker(A_s)) \leq dim(X/A_s(X))$. Assume the opposite. Then, as $ker(A_s)$ is finite-dimensional, both subspaces are complemented and there are decompositions $X = ker(A_s) \oplus E = A_s(X) \oplus G$ for closed subspaces E and G. Let p be the corresponding continuous projection to $ker(A_s)$. Now if, as assumed, $dim(ker A_s) > dim G$, there is a linear noninjective but surjective map $f : ker A_s \rightarrow$, and we define $F := A + f \circ p$, this is continuous, linear and even compact, as f has finite-dimensional image. Now we have $F - s\mathbf{1} = A_s + f \circ p$ and thus

$$(F - s\mathbf{1})(E) = S(E). \tag{1}$$

On the other hand, we have

$$(F - s\mathbf{1})|_{ker(A_s)} = f|_{ker(A_s)}. \tag{2}$$

As $f(ker A_s) = G$, by Equations 1 and 2 we get $(F - s\mathbf{1})(X) = A_s(X) + G = X$. But if we apply Equation 2 to a vector $x_0 \neq 0$ with $f(x_0) = 0$ which exists because of noninjectivity of f, it turns out that s is an eigenvector of F, therefore $F - s\mathbf{1}$ cannot be surjective, a contradiction. The other direction $dim(ker(A_s)) \geq dim(X/A_s(X))$ is left as an **exercise**. □

Theorem 4.18 *Let X, Y be Banach spaces, $A_i \in BL(X, Y)$ with $A_i(X)$ finite-dimensional, and let $A_i \rightarrow A \in BL(X, Y)$. Then A is compact.*

Proof: Exercise. □

If the Banach space has a Schauder basis, also the converse is true:

Theorem 4.19 (Approximation property) *Let X, Y be Banach spaces, X with Schauder basis, let $A \in BL(X, Y)$ be compact, then there are $A_i \in BL(X, Y)$ with $A_i(X)$ finite-dimensional and $A_i \rightarrow A$.*

Therefore, as every Banach space is metrizable, in a Banach space with Schauder basis the compact operators are exactly the topological closure of the operators with finite-dimensional range.

Exercise: Let X be a Banach space over $\mathbb{K}$. Let $A : X \rightarrow X$ be a compact operator and $\lambda \in \mathbb{K} \setminus \{0\}$. Prove that the equation

$$Af - \lambda f = g$$

has either a unique solution $f \in X$ for every $g \in X$ or there are some $g_1, g_2 \in X$ with no solution whatsoever for g_1 and infinitely many solutions for g_2.

Exercise: Let (Ω, μ) be a σ-finite measured space, and let $\mu^2 := \mu \times \mu$ be the product measure on $\Omega \times \Omega$, and $K \in L^2(\mu \times \mu)$. Define, for $f \in L^2(\mu) =: X$,

$$A_K f(s) := \int_\Omega K(s,t) f(t) d\mu(t).$$

(a) Prove that $A \in B(X, X)$ with $||A|| \leq \sqrt{\int_{\Omega \times \Omega} |K(s,t)|^2 d\mu(s) d\mu(t)}$.
(b) For any $a_i, b_i \in X$, $i = 1, ...n$, define $K(1)(s,t) := \sum a_i(s) b_i(t)$. Show that this is an element of $L^2(\mu \times \mu)$ and that $A_{K(1)}(X)$ is n-dimensional.
(c) Show that for every K , A_K is compact.
(d) Now apply this to the case $(\Omega, \mu) := ([0,1], dt)$ and $K = 1$ and show that the equation

$$\int_0^s f(t) dt + \lambda f(s) = g(s)$$

has a solution for every $g \in L^2([0,1])$.

Exercise: Read now the primer about the Inverse Function Theorem in Banach spaces by Ralph Howard!